QUASI-ONE-DIMENSIONAL SOLUTIONS OF THE EQUATIONS OF A HIGH-CURRENT ELECTRON BEAM

V. N. Danilov

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On the basis of one-dimensional solutions of the equations of an axisymmetric, double-flow beam, an adiabatic approximation is constructed that makes it possible to describe the effect of a slightly inhomogeneous magnetic field on a stationary, quasineutral beam. Tubular and stratified beam configurations, confined near the axis, at currents of the order of the critical current and substantially in excess of the critical current, are considered.

Equilibrium states of a high-current, relativistic beam of electrons [1] in a filled plasma tube have been studied in [2]. Within the framework of the adiabatic approximation we consider here the problems of control by the parameters of a stationary axisymmetric beam in a tube with the aid of an external, slightly inhomogeneous magnetic field. This field makes it possible effectively to change the relationship between velocity components of the electrons, the beam radius, etc. Double-flow beams, being more regulated so far as velocities are concerned, are of interest for the further transformation of energy. The adiabatic approximation for a double-flow beam can be constructed from a solution that is one-dimensional with respect to the radius r, the constants of this solution being considered as quasiconstants, depending slightly on the longitudinal coordinate z, with the exception of integrals of the motion: the electron energy $\mathscr{E}$, the azimuthal component P_θ of the generalized momentum, the adiabatic invariant w, and the current J. A quasi-one-dimensional double-flow beam consists of two subcurrents of electrons, which differ from each other only in the sign of the radial component of the 4-velocity,

$$u_r = \pm(\mathscr{E}^2 - 1 - r^{-2}u_\theta^2 - u_z^2)^{1/2}, \quad u_{\theta(z)} = A_{\theta(z)} + P_{\theta(z)} \tag{0.1}$$

and it is bounded by the surfaces r_+, r_- on which these subcurrents become converted from one type to the other [3]:

$$r = r_\pm, \ u_r = 0, \ w = \int_-^+ |u_r|\,dr, \quad J = 2\pi \int_-^+ \rho u_z r\,dr \tag{0.2}$$

Here (O, A_θ, A_z) is the vector potential of the self-consistent field (O, H_θ, H_z)

$$H_\theta = -rA_{z,r}, \ H_z = r^{-1}A_{\theta,r} \quad H^2 \equiv r^{-2}H_\theta^2 + H_z^2$$
$$r(r^{-1}A_{\theta,r})_{,r} = 2\pi\rho u_\theta, \ r^{-1}(rA_{z,r})_{,r} = 2\pi\rho u_z, \ r\rho|u_r| = 2I \tag{0.3}$$

the potential of the electric field is omitted, in keeping with the neutrality of the beam; ρ is the scalar electron density; the quasiconstant I has the sense of a rotary current formed by oscillations of the electrons that are transverse to the beam; an index after a comma denotes a derivative with respect to the corresponding coordinate; the azimuthal components are defined as covariant; and the physical constants e, m, c are omitted, which corresponds to departures from the usual notation as indicated by the arrows:

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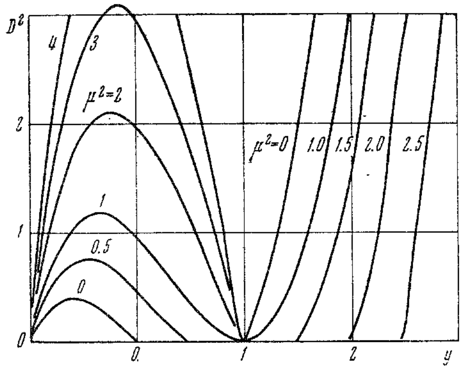

Fig. 1

$$
\begin{gathered}
P/mc \to P,\ \mathscr{E}/mc^2 \to \mathscr{E},\ eA/mc^2 \to A \\
eH/mc^2 \to H,\ 2e^2\rho/mc^2 \to \rho,\ -2eJ/mc^3 \to J
\end{gathered} \tag{0.4}
$$

The double-flow quality of a beam requires identity of the electrons with respect to the constants P_θ and w and is achieved by injection of a very narrow beam with identical electron velocities onto a diaphragm or, for instance, under the following conditions: the cathode in the vacuum part of the accelerator lies on the magnetic surface $A_\theta = -P_\theta$; the region of the single-flow stream at the cathode is surrounded by the outermost trajectory and adjoins the double-flow beam on it, the latter being everywhere in a state where conditions are adiabatic and penetrating smoothly to the diaphragm at a small angle with the boundary. Violation of these conditions results in velocity dispersion. In this case the double-flow beam considered below will correspond to an actual one as a model that takes account of all qualitative properties with the exception of the detailed structure of the distribution of the magnetic self-field along the radius.

1. Tubular Beam

The system of equations (0.1), (0.3) can be reduced to two by simple substitutions

$$
\begin{gathered}
v_{,\sigma\sigma} - v\psi_{,\sigma}{}^2 = e^{\varepsilon\sigma} v/2u + \varepsilon^2 v \sin^2\psi,\ u = (1 - v^2)^{1/2} \\
(\psi_{,\sigma} v^2)_{,\sigma} = \varepsilon^2 v^2 \sin\psi\cos\psi,\ r \equiv Re^{\varepsilon\sigma},\ \varepsilon^2 8\pi IR \equiv \varkappa \\
u_z + ir^{-1}u_\theta \equiv \varkappa v e^{i\psi},\ u_r \equiv \pm\varkappa u,\ \varkappa \equiv (\mathscr{E}^2 - 1)^{1/2},\quad i \equiv \sqrt{-1}
\end{gathered} \tag{1.1}
$$

For small ε the beam is confined close to the quasicylinder R(z), and the solution can be expanded in powers of ε, converting to the short coordinate $s \equiv r - R$

$$
s = \varepsilon R\,(\sigma + {}^1/_2\,\sigma^2\,\varepsilon),\ \sigma \equiv \pm\,\sigma' + \varepsilon\sigma'' \tag{1.2}
$$

Integrating (1.1) with an accuracy up to ε^2, we easily obtain

$$
\sigma_{,v}{}' = v/\delta,\ \delta \equiv [(u - u_1)(u_2 - u)(u_3 - u)]^{1/2} \tag{1.3}
$$

$$
(rH_z + iH_\theta)\varkappa^{-1} = D(\varepsilon\cos\vartheta)^{-1}\exp i(\psi \mp \vartheta) + v\sin\psi + ie^{i\psi}\sigma_{,v}{}''(\sigma_{,v}{}')^{-2},\ \sigma_{,v}{}'' = {}^1/_2(\sigma_{,v}{}')^3\quad (u\sigma' - w')
$$

$$
\psi = \Phi \pm \psi' + \varepsilon\psi'',\ \psi_{,\sigma}{}' = Dv^{-2},\ \mathrm{tg}\,\vartheta \equiv \delta/D
$$

Here Φ, D, μ are quasiconstants and u_1, u_2, u_3 are real roots of the cubic equation, indicated in Fig. 1,

$$
(1 - y^2)\ (\mu^2 - y) = D^2 \leqslant \mu^2,\ u_1 \leqslant 0 \leqslant u_2 \leqslant 1 \leqslant u_3 \tag{1.4}
$$

Finally, the solution is expressed in terms of elliptic integrals of the first F, the second E, and the third kind Π:

$$
\begin{gathered}
\sigma' = 2(u_3 - u_1)^{-1/2}[u_3F(\varphi, k) - (u_3 - u_1)E(\varphi, k)]\,\Big|_\varphi^{\pi/2} \\
u = u_1 + (u_2 - u_1)\sin^2\varphi,\quad k^2 \equiv (u_2 - u_1)/(u_3 - u_1) \\
\psi' = \frac{D}{\sqrt{u_3 - u_1}}\left[\frac{\Pi(n_1, k, \varphi)}{1 - u_1} - \frac{\Pi(n_2, k, \varphi)}{1 + u_1}\right]\Big|_\varphi^{\pi/2} \\
n_1 = -(u_2 - u_1)/(1 - u_1),\ n_2 = (u_2 - u_1)/(1 + u_1) \\
w' \equiv \int_0 u\,d\sigma' = {}^2/_3(u_3 - u_1)^{-1/2}\{[3u_3{}^2 - (u_3 - u_1)(u_3 - u_2)]\,F(\varphi, k) - \\
- 2\mu^2(u_3 - u_1)E(\varphi, k) + {}^1/_2(u_3 - u_1)(u_2 - u_1)\sin 2\varphi\,(1 - k^2\sin^2\varphi)^{1/2}\}\,\Big|_\varphi^{\pi/2}
\end{gathered} \tag{1.5}
$$

The functions marked with a dash are reckoned from the line of maximum $u = u_2$, which is situated almost at the center of the beam.

The upper sign corresponds to the upper half of the beam, while the lower sign corresponds to the lower half. The two-valued representation separates the sheets of the required function s(v), and this

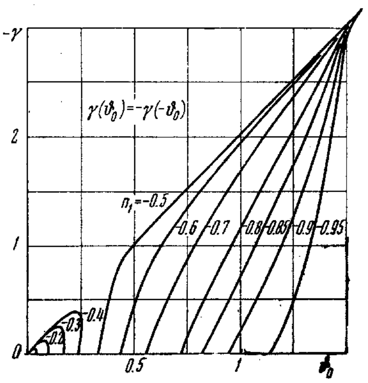

Fig. 2

enables one to avoid nonuniformity of the asymptotic representation in the neighborhood of the branch point $u = u_2$. Denoting values of functions of fixed sign on the boundaries (0.2) by the index zero, we can write

$$
\begin{gathered}
s_\pm = \pm a_0 + \ldots,\ a_0 = \varepsilon\sigma_0' R,\ \psi_\pm = \Phi \pm \psi_0' + \varepsilon\psi_0'' \\
\varphi_0 = \arcsin\,[-u_1/(u_2 - u_1)]^{1/2},\quad \mu = D/\cos\vartheta_0 \\
-\frac{J}{\varkappa} = \frac{\mu}{\varepsilon} 2\cos\Phi \sin\frac{\gamma}{2} + \frac{w_0'}{\delta}\sin\Phi\,\sin\psi_0' \\
w = 2\varepsilon R\varkappa w_0',\ \gamma = 2\,(\psi_0' - \vartheta_0),\ \gamma\,(\vartheta_0) = -\gamma\,(-\vartheta_0)
\end{gathered}
\tag{1.6}
$$

The values of σ_0', ψ_0', w_0' are obtained by replacing φ with φ_0 in expressions (1.5). The signs of μ, D, ϑ_0 are the same and opposite to the sign of γ, the "angle" ϑ_0 varies in the range $-\pi/2, \pi/2$. During passage from the lower to the upper boundary the moduli of the vectors $(r^{-1}u_\theta, u_z)$, $(r^{-1}H_\theta, H_z)$ remain almost the same, and in the (z, r, θ) plane these vectors turn through the angles $2\psi_0'$ and γ, respectively. The solution obtained above is determined by two parameters: μ, D or n_1, ϑ_0. In Fig. 2 the dependence of the angle γ through which the magnetic field turns on the parameters n_1, ϑ_0 is represented. In particular, Figs. 1 and 2 enable one to calculate the state of a high-current beam, as determined below (2.4).

Values of the parameters n_1, n_2 are confined to the strip

$$
\begin{gathered}
0.5 < -n_1 < 1,\ -n_1/(1+n_1) \equiv N_1 < n_2 < \infty \\
0 < -n_1 < 0.5,\ N_1 < n_2 < -n_1/(1+2n_1) \equiv N_2
\end{gathered}
\tag{1.7}
$$

At the boundaries of this strip the solution has the following asymptotic forms.

The case $n_2 \approx N_1$ corresponds to a very intense external field

$$
\begin{gathered}
u_2 \approx -u_1 \approx n/(2-n) = |\sin\vartheta_0|,\ n \equiv -n_1 \\
\psi_0' \approx \vartheta_0,\ \varphi_0 \approx \pi/4,\ \gamma \approx k \approx 0,\ u_3 \approx \mu^2 \gg 1
\end{gathered}
\tag{1.8}
$$

The case $n_2 \approx N_2$ corresponds to a small transverse velocity (cold beam)

$$
\begin{gathered}
\sigma' = {}^1/_2\,|\mu|^{-3}(\sin\tau/\cos\tau_0 - \tau),\quad u = |\mu|\,\sigma_{,\tau}' \\
u' \equiv 2u_2/|\gamma| = (\tau_0\cos\tau_0)^{-1} - \tau_0^{-1},\quad 0 \leqslant \tau \leqslant \tau_0 \\
\operatorname{tg}\tau_0 = 2\mu^2|\vartheta_0|,\quad J\varepsilon\,(\varkappa\cos\Phi\,|\mu|)^{-1} = |\gamma| \ll 1 \\
\mu^2|\gamma| = \tau_0,\ a' \equiv 2a_0|\mu|\,(\varepsilon R|\gamma|)^{-1} = \operatorname{tg}\tau_0/\tau_0 - 1 \\
\Gamma \equiv 2\,|\mu|\,w\,(\varepsilon R\varkappa\gamma^2)^{-1} = \tau_0^{-2}\,[(1 + {}^1/_2\cos^{-2}\tau_0) - {}^3/_2\operatorname{tg}\tau_0]
\end{gathered}
\tag{1.9}
$$

For small μ^2 a cold beam is described more simply

$$
\begin{gathered}
u_2 \approx \mu^2 - D^2 \ll 1,\ \sigma' = (u_2 - {}^1/_{12}t^2)\,t,\ u = \sigma_{,t}' \\
t_0 = 2u_2^{1/2},\ w_0' = {}^{16}/_{15}u_2^{5/2},\ \sigma_0' = {}^4/_3 u_2^{3/2},\ \gamma \approx -2\vartheta_0
\end{gathered}
\tag{1.10}
$$

The case $n_2 \to \infty$ corresponds to small D^2. In the region $\mu^2 < 1$ the solution has the form

$$
\begin{gathered}
u_1 \approx -1,\ u_2 \approx \mu^2 + D^2(\mu^4 - 1)^{-1},\ u_3 \approx 1 \\
k^2 = {}^1/_2(\mu^2 + 1),\ \vartheta_0 = -\gamma/2 = \pm\pi/2 \\
\sigma_0' = \sqrt{2}\,(F - 2E)\,\Big|_{\varphi_0}^{\pi/2},\ v_{\min} = (1 - \mu^4)^{1/2} \\
w_0' = {}^1/_3\sqrt{2}\,[(2\mu^2 + 1)\,F - 4\mu^2 E]\,\Big|_{\varphi_0}^{\pi/2} - {}^2/_3\,|\mu|
\end{gathered}
\tag{1.11}
$$

In this beam the transverse velocity is large (hot beam), but the longitudinal velocity is still considerable. In contrast to the preceding, for $\mu^2 > 1$ the solution corresponds to a rotation of the velocity through nearly 180° and a small rotation of the magnetic field:

$$
\begin{gathered}
u_1 \approx -1,\ u_2 \approx 1 - {}^1/_2 D^2(\mu^2 - 1)^{-1},\ u_3 \approx \mu^2 \\
\sigma_0' = 2\,(1 + \mu^2)^{-1/2}\,[\mu^2 F - (1 + \mu^2)\,E]\,\Big|_{\pi/4}^{\pi/2} \\
w_0' = {}^2/_3\,(1 + \mu^2)^{-1/2}\,[(2\mu^4 + 1)\,F - 2\mu^2(\mu^2 + 1)E]\,\Big|_{\pi/4}^{\pi/2} - {}^2/_3\,|\mu|, \\
-D/\gamma \equiv D' \simeq \{(2\mu^2 - 2)^{-3/2}\,[{}^{43}/_{16} + (\mu^2 - 1)^{-1} + (2\mu^2 - 2)^{-2}] - 2/|\mu|\}^{-1}, \\
k^2 \simeq 2\,(1 + \mu^2)^{-1}
\end{gathered}
\tag{1.12}
$$

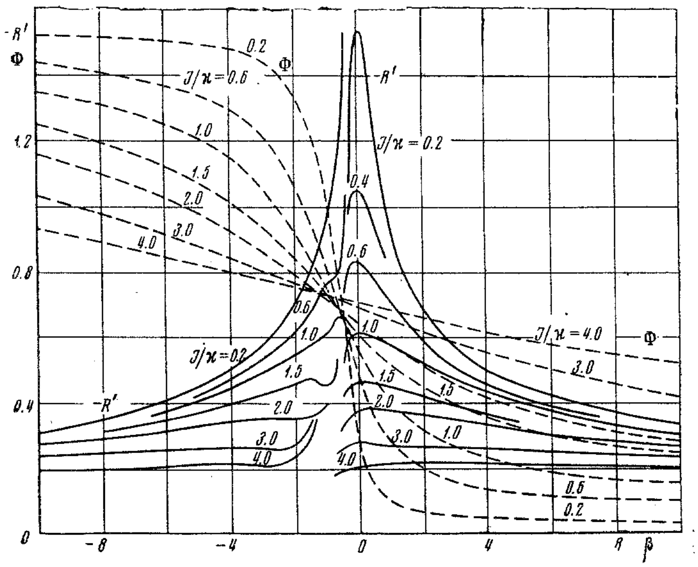

Fig. 3

In this drifting beam the longitudinal velocity at the axis is small. The current in the upper half of the beam almost compensates the current in the lower half, but, despite the fact that the total current is small, the drop in magnetic pressure toward the center of the beam is large. The immediate neighborhood of the singular point $\mu = 1$, $D = 0$, in which reflection or a branching of the beam with a bend in the axis can occur, is reached because of the conservation of w in a very strong external field.

The solution for a wide beam could be constructed numerically, using the following local solution of the original equations (1.1):

$$\begin{gathered}
\sigma = \sigma_n + \sigma', \quad v = v_n + v', \quad r = Re^{\sigma}, \quad \varepsilon = 1 \\
\sigma' = \omega_n^{-1} P_n \sin\tau - e_n \omega_n^{-2}(1 - \cos\tau) - \alpha_n \omega_n^{-3}(\tau - \sin\tau) \\
v' = 1 - v_n - {}^1/_2 \omega_n^2 (\sigma_{,\tau}')^2, \quad \omega_n^2 \equiv D^2 v_n^{-3} + v_n \sin^2\psi_n, \\
\alpha_n \equiv v_n [2(1 + v_n)]^{-1/2} \exp \sigma_n \\
v'|_{\tau=0} = \sigma'|_{\tau=0} = 0, \quad v_{,\sigma}|_{\tau=0} = e_n, \quad P_n = [2(1 - v_n)]^{1/2}
\end{gathered} \tag{1.13}$$

It is not difficult to supplement (1.13) with the succeeding term of the expansion in the small dashed increments. Then the original equations can be integrated for a wide beam with large, uniform steps in σ, $n = 1, 2, \ldots$, right up to the boundaries, on which the essential singularities of the problem are located.

The asymptotic representation (1.13) removes these singularities, and does so uniformly with respect to all the parameters. For small ω_n the expansion is made according to powers of τ.

2. Adiabatic Equations

The equations for the quasiconstants ε, μ, D, Φ, R follow from the conservation of w and J, defined in (1.6), the absence of H_θ inside the beam, the continuity of A_θ on the inner boundary r_-, and the presence of an axial field B(z) outside the beam:

$$\begin{gathered}
C(\Phi - \gamma/2) = \lambda \cos\psi_-, \quad \lambda \equiv \omega_0'/2\delta_0, \quad R' \equiv -R\varkappa/2P_\theta \\
\mu/\varepsilon \equiv C = (1 + \varepsilon\sigma_0')/R' + (1 - \lambda)\sin\psi_-, \quad \beta \equiv -2P_\theta B\varkappa^{-2} \\
C\cos(\Phi + \gamma/2) = R'(1 + \varepsilon\sigma_0')\beta - (1 + \lambda)\sin\psi_+
\end{gathered} \tag{2.1}$$

Of interest are an electron current in which the velocities have maximum order [cold beam (1.9), (1.10)], and also a current of identical oscillators with large energy of the transverse oscillations [hot beam (1.11), (1.12)]. Solutions of the adiabatic equations, constructed below, apply mainly to these states.

In a weak external field ($\beta \sim 1$) the beam is cold* ($\mu^2 \sim u_2 \sim \varepsilon_2$). From (1.6), (1.10), (2.1) it follows that

* The adiabatic approximation is applicable under the condition $u_2 \gg a_0/L \gg 1$, where L is the scale of the periodicity of B. Accordingly u_2 is bounded from below.

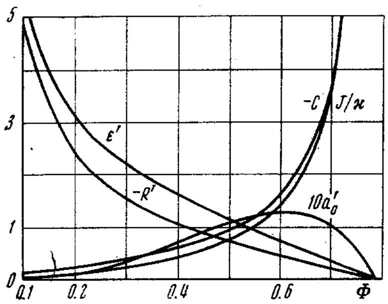

Fig. 4

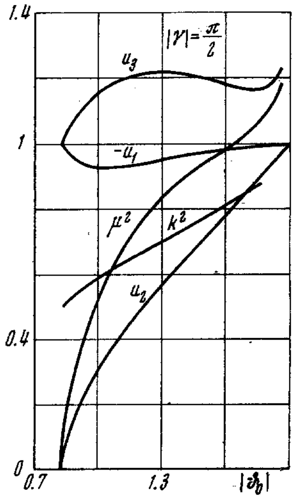

Fig. 5

$$\beta R' = \sin\Phi\cos^2\Phi\{1 - [1 + \beta\cos 2\Phi\sin^{-2}\Phi\cos^{-4}\Phi]^{1/2}\}$$
$$C = (R'\beta - \sin\Phi)/\cos 2\Phi = -J(\varkappa\sin 2\Phi)^{-1}$$
$$\varepsilon' \equiv 2\varepsilon\,|P_\theta|^{1/6}(15w)^{-1/6} = |R'(C\sin\Phi)^5|^{-1/6} \quad (2.2)$$
$$a' \equiv {}^3/_4\varkappa\,({}^{15}/_{32}w)^{-2/3}|2P_\theta)^{-1/3}a_0 = \varepsilon'^4|R'(C\sin\Phi)^3|$$

Negative values of β correspond to a change in sign of the external field B on the way from the cathode and a solution in the region $-\beta > \sin^2\Phi$ with the opposite sign in front of the radical in R'. In Fig. 3 the solid curves depict the variation of the dimensionless radius R' of the beam with the external field parameter β, while the dashed curves depict that of the angle Φ. Monotonic variation of a positive β results in a homogeneous beam. The dependence of parameters of the homogeneous state on the "winding" angle of the trajectory is depicted in Fig. 4 and is determined by the expressions

$$R' = -\mathrm{ctg}\,2\Phi/\cos\Phi, J/\varkappa = \sin\Phi\,\mathrm{tg}\,2\Phi \quad (2.3)$$

In the case of strong fields ($\beta \sim \varepsilon^{-2}$) and large currents ($J \sim \varkappa\varepsilon^{-1}$) it follows from (1.6) and (2.1) that, with an accuracy to ε,

$$\Phi = \gamma/2,\ \cos\gamma = {}^1/_2\alpha\,[(1 + 4\alpha^{-2})^{1/2} - 1],\ \alpha\beta \equiv (J/\varkappa)^2 \quad (2.4)$$
$$R' = \pm(\cos\gamma/\beta)^{1/2},\ \varepsilon = (w\,|\mu|)^{1/2}(4\,|P_\theta|\,w_0')^{-1/2}$$

With a decrease in B the beam becomes homogeneous, with parameters determined by the graphs of Fig. 5 and the relations

$$\gamma \approx 2\Phi \approx \pm\pi/2,\ R = |2P_\theta|/J,\ C = \mp J/\varkappa \quad (2.5)$$

The quantity $-2\pi P_\theta$ is equal to the magnetic flux surrounding the emitting portion of the cathode. Therefore the sign of the quantity R' in (2.4) must be the same as the sign of the field at the cathode. The state (2.5) is determined by the equilibrium of the magnetic pressure of the internal axial and the external azimuthal fields. The drop from the boundary of the beam to the center is compensated by the density of the transverse momentum flux, which is equal to the energy density of the transverse oscillations

$$H^2 \approx \varkappa^2\mu^2(\varepsilon R)^{-2} \approx {}^1/_4 J^4 P_\theta^{-2},\ \Delta H^2 = u_2\varkappa^2(\varepsilon R)^{-2} \quad (2.6)$$

For currents of the order of the critical current ($J \sim \varkappa$) the magnetic field either rotates hardly atall

$$\gamma C \approx -J/\varkappa,\ \ C = \pm\beta^{1/2},\ R = (-2P_\theta/B)^{1/2}, \Phi \approx \gamma/2 \quad (2.7)$$

or it rotates through an angle of the order ± 180°

$$\gamma \approx \pm\pi,\ \vartheta_0 \approx \pm\pi/2,\ C = -RB/\varkappa,\ B = (2P_\theta/B)^{1/2} \quad (2.8)$$

The latter case corresponds to the asymptotic form (1.11). The external field must change sign on the way from the cathode. The azimuthal currents are large, and the magnetic field is mainly axial. Figure 6 shows the evolution of the beam in its dependence on B':

$$^1/_2\,|B|\,w\varkappa^{-2} \equiv B' = |\mu|\,w_0',\ \ a'_0 \equiv 2a_0\varkappa/w = \sigma_0'/w_0' \quad (2.9)$$

In the case (2.7) the equation $\gamma = 0$ results in the asymptotic form for a hot beam (1.8), (1.12). The evolution of a drifting beam (1.12) is shown in Fig. 7, where a_0', B' are defined by (2.9)

$$2(\varkappa/J)\,|P_\theta|^{1/2}w^{-1/2}|D| \equiv D'' = D'(B')^{-1/2} \quad (2.10)$$

This beam does not exist for a field B' less than $\pi/4$. Near this point the asymptotic form (1.12) and the adiabatic conditions break down. The third solution of the equation $\gamma = 0$

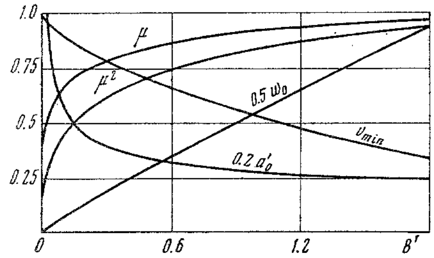

Fig. 6

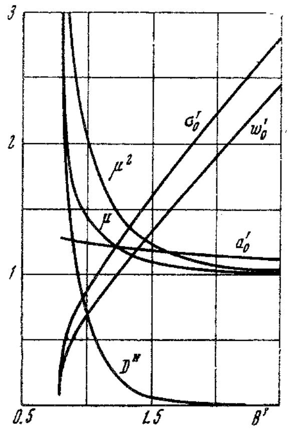

Fig. 7

$$\sin^2\vartheta_0 = (x-1)(x^2-2x+2)(x+1)^{-1}x^{-2},\quad 1<x<1.5$$
$$u_{1(2)} = 1 - x(\mp)|\sin\vartheta_0|x \tag{2.11}$$
$$u_3 = (x^2-2x+2)(x^2-1)^{-1},\quad \mu^2 = x^2(3-2x)(x^2-1)^{-1}$$

departs only slightly from the asymptotic form for a cold beam (1.9). Figure 8 shows the functions

$$\frac{(J\varkappa)^{1/2}a_0}{(|P_\theta|w^3)^{1/4}} \equiv a_0' = a'\Gamma^{-3/4},\quad \frac{2|wP_\theta|^{1/2}}{J\varkappa}|B| \equiv B' = \Gamma^{1/2} \tag{2.12}$$
$$2u_2(\varkappa/J)^{1/2}|P_\theta|^{1/4} - w^{-1/4} \equiv u_2' = u'\Gamma^{-1/4}$$

describing the evolution of a beam in conformity with (2.7), (1.9).

3. Stratified Beam

A stratified beam can be constructed from narrow tubular beams (in the terminology [2] of current filaments), if the parameters of the beam described above are considered to depend on the number m of the layer and the field strength and the magnetic potential match on the boundary of neighboring layers

$$H_z^+ - H_z^- = H_{z,m}^-,\quad A_\theta^+ - A_\theta^- + H_z^+(r^+ + {}^1/_2 d)d = A_{\theta,m}^- \tag{3.1}$$
$$H_z^\pm = (\varkappa/r\pm)\{C\cos(\Phi \pm \gamma/2) + (1+\lambda)\sin(\Phi \pm \psi_0')\}$$
$$H_\theta^+ - H_\theta^- = H_{\theta,m}^-,\quad r^+ - r^- + d = r_{,m}^- = r^-(m+1) - r^-(m) \tag{3.2}$$

Equations (3.1), together with the conservation of J(m), w(m) (1.6) and conditions of type (2.1) on the outer layers

$$m=1,\ H_\theta^- = 0,\ {}^1/_2 H_z^- r_-^2 = A_\theta^-,\ m = M,\ H_z^+ = B \tag{3.3}$$

constitute a boundary problem for the parameters R, Φ, μ, D, ε and the gap d between the layers. Below we consider beams with a large number of layers M and a smooth dependence of the parameters on the layer number, as is reflected in (3.2). Then Eqs. (3.2) take the form

$$H_\theta^-/\varkappa \doteq C\sin(\Phi - \gamma/2) - \lambda\cos(\Phi - \psi_0') \equiv -S$$
$$R_{,m} = 2a_0 + d,\quad S \equiv \int_1^m J/\varkappa dm,\quad C \equiv \mu/\varepsilon \tag{3.4}$$

For a drifting beam (1.12) Eqs. (3.1), with an accuracy to ε, become

$$\nu 2R\cos\Phi = hd - P_{\theta,m}/\varkappa,\quad \nu \equiv \operatorname{sign} C,\quad -h \equiv C\cos\Phi$$
$$R(h/R)_{,m} - 2(\sigma_0'\mu - \nu)\cos\Phi + 2\lambda\nu\cos\Phi = \gamma C\sin\Phi \tag{3.5}$$

In the case $P_{\theta,m}/R \ll 1$ Eqs. (3.5) become expressions for the small angles γ, $\pi/2-\Phi$. This is a homogeneous state. The required external field is determined from (3.3) and is proportional to $P_{\theta,m}(M)$. For the densest packing of the layers (d = 0) the following current distribution follows from (1.6), (3.4), and the relation $\mu^2 a_0' \approx 1$ (see Fig. 6):

$$S = \lambda(1)[R/R(1)]^{1/2}$$
$$J = \lambda(1)[RR(1)]^{-1/2}w \tag{3.6}$$

The total current $\varkappa S(M)$ can be very large.

For a beam with cold layers (1.9), (1.10) small values of γ are necessary, and Eqs. (3.1) take the form

$$(R\sin\Phi - P)_{,m} + hd = 2R(\psi_0'\cos\Phi + \varepsilon\sigma_0'\sin\Phi)$$
$$R(h/R)_{,m} = 2(\mu\sigma_0' - \psi_0')\cos\Phi + (2\varepsilon\sigma_0' + C\gamma)\sin\Phi \tag{3.7}$$
$$-h \equiv C\cos\Phi + \sin\Phi \simeq RH_z^\pm/\varkappa,\quad P \equiv P_\theta/\varkappa$$

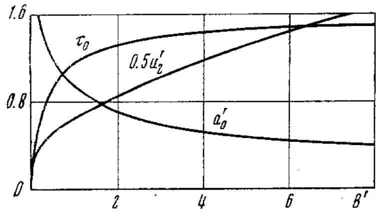

Fig. 8

If the axial field inside or outside the field is large, the terms ψ_0', $\varepsilon\sigma_0'$ can be neglected:

$$h \approx P_{,R}, \quad R\,(P_{,R}/R)_{,R} + S_{,P}S = 0 \tag{3.8}$$

The function S(P) is determined by conditions at the cathode. Elementary cathodes P = const must be placed at various levels of the magnetic flux. As a whole, the cathode is intersected by the magnetic lines of force. An example of a solution

$$S = \{k_* [P - P(1)]\}^{1/2}, \quad P = {}^1/_4 k_* [R^2 - R^2(1) + 2R^2 \ln R(M)/R(1)] - {}^1/_2 R^2 B/\varkappa, \quad k_* = \text{const} \tag{3.9}$$

shows that the current $\varkappa S$ can be large for a large field at the cathode. With a decrease in the external field the beam broadens somewhat.

In a beam with opposed azimuthal currents the quantities μ, σ, Φ, γ, ψ_0', A_θ, H_z, and P_θ change sign from one layer to the next. A conversion to the representation

$$\mu \to (-1)^m \mu, \quad \vartheta \to (-1)^m \vartheta, \ldots, \quad P_\theta / \varkappa \to (-1)^m P \tag{3.10}$$

results in quantities with a constant sign and a change in signs in front of H_z^-, A_θ^- in Eqs. (3.1):

$$\begin{gathered} {}^1/_2 R\,(H_z^-/\varkappa)_{,m} + C\cos\Phi\cos\gamma/2 + (1+\lambda)\sin\Phi = \\ = -\mu\sigma_0'\sin\Phi\sin\gamma/2, \quad 2\varkappa\,(R\sin\Phi - P) = -dH_z^+R \end{gathered} \tag{3.11}$$

For a cold, rarefied ($a_0 \ll d$) beam the complete system of equations is

$$\begin{gathered} Rh\,(h/R)_{,m} + SS_{,m} = 2h\sin\Phi, \quad hR_{,m} + 2R\sin\Phi = 2P \\ \operatorname{tg}\Phi = 2\varkappa h/J, \quad \operatorname{tg}\gamma/2 = h/S, \quad \pm H_z^{\pm}R \approx \varkappa h \\ h(1) = 2P/R - \sin\Phi, \quad h(M) \approx (-1)^M BR/\varkappa - \sin\Phi \end{gathered} \tag{3.12}$$

Two solutions are given below that correspond to a powerful, localized beam under the conditions

$$S(M) \gtrsim M \gg 1, \quad |BP(M)| > (2\text{–}3)\varkappa \tag{3.13}$$

The first condition ensures the compensation of the centrifugal forces by the pinch forces and equilibrium. Violation of the second condition results in a rapid growth in the radius of the outer layers. Within the scope of (3.13) the angle Φ is close to $\pi/2$

$$\begin{gathered} S = (\Pi_* m)^{1/2}, \quad P = P_*, \quad R = P_*(1 - ke^{-2f}) \\ hP_* = Re^{2f}[U - (\Pi\varkappa/2k)\ln(R/R(M))], \quad R(1) = P_*(1-k) \\ m - 1 = \int_0 h\,df \approx {}^1/_2[h - h(1) + \Pi_* f], \quad U = 2(1-k)^{-2} \approx \\ \approx (\Pi_*/2k)\ln(R(M)/R(1)) - PB\varkappa^{-1}\exp - 2f(M) \end{gathered} \tag{3.14}$$

Asterisks denote quantities that are chosen as constants. In the case

$$\begin{gathered} f(M) > 2 \text{ and } |B| \approx (2-3)\,\varkappa/P_* \\ M \approx {}^1/_2\Pi_* f(M) - (1-k)^{-1}, \quad S(M) = \Pi_* \sqrt{f(M)} \\ 1 - k = [(\Pi_*/8)\ln(\Pi_*/8)]^{-1/2} \ll 1 \end{gathered} \tag{3.15}$$

In the second solution the currents in the various layers are identical:

$$\begin{gathered} J/\varkappa = (1 + \omega_*^2)^{1/2}, \quad R = P_*(1 - ke^{-2\,\cdot}), \quad P = P_* \\ hP_* = RUe^f(\omega_*^{-1}\sin\omega_* f + \cos\omega_* f), \quad m_{,f} = h \end{gathered} \tag{3.16}$$

For small ω_* and minimum B we obtain

$$U \approx 2(1-k)^{-2}, \quad S(M) \approx M = U\omega_*^{-2}\exp\pi/2\omega_*$$

Here, as in (3.14), the modulus of the axial field, starting with the already large value U/P_* inside the beam, grows exponentially, but farther on a steep drop almost to zero at the outer layer occurs. It is possible that the self-field of such a beam is able to provide the magnetic flux $(-1)^{m+1}\, 2\pi P\varkappa$ which is small, but of variable sign, in the region outside the cathode, necessary for the formation of opposed azimuthal currents.

We can now put some of the results in concrete form, as estimates of the dimensional quantities in (0.4). Suppose that an annular knife edge of radius R_k, directed along a magnetic line of force, serves as the cathode of the system in [1]

$$P_\theta = -B_k R_k^2 e/2c = \text{const}$$

Suppose the reverse currents have dissipated and the self-field of the beam is commensurable with or exceeds the external field B.

For large external fields at the cathode $B_k \sim 1.7\,(\varkappa/a)$ kOe · cm and large currents $J \sim \varkappa(R_k/a)\,8.5$ kA the beam assumes a tubular form with thickness $a \ll R_k$. If the beam is extracted smoothly from the external field, the beam does not change its structure very much, equilibrating itself with its self-fields (internal axial and external azimuthal) that are commensurable with B_k. On the average the trajectories wind about a cylinder of radius $c^2P_\theta/eJ \sim R_k$ under an angle of 45°, and the density of the oscillator energy remains commensurate with that of the translational and rotational energies. By decreasing the current to $J \sim 8.5\,\varkappa$ kA one can obtain the drift beam (1.12) (Fig. 6), in which practically all the linear energy density $\sim \varkappa^2(R/a)$ J/cm is due to the transverse oscillators and is commensurable with the energy density of a high-current beam. Such a beam is of interest as a maximally dense current of strongly nonlinear oscillators for obtaining radiation. However, a drifting beam breaks down with fields that are substantially less than $1.7\,(\varkappa/a)$ kOe · cm. Passage through the diaphragm must result in an increase in the oscillator fraction of the energy. A further increase in the external field in low-current beams can adiabatically convert translational and rotational energy into oscillator energy (Fig. 5).

In a weak external field $B_k \sim 1.7\,(\varkappa/R_k)$ kOe · cm the beam will be a narrow tube with small oscillator energy if it penetrates the diaphragm at a small angle to the external field. Upon extraction from the external field the radius of a cold beam increases markedly (Figs. 3 and 4). In a cold beam the centrifugal forces exert a substantial counteraction to pinch compression. Therefore the azimuthal external field is markedly larger than the axial internal field, there is an increase in the azimuthal currents, and, associated with this, the increase in the centrifugal forces results in a sharp increase in the limiting currents (Fig. 4). If a concentric tubular beam with opposed azimuthal currents is formed, then for maximum centrifugal forces the axial fields will be almost compensated. The external field required for localization of a stratified beam within a radius R was found to be not large, $1.7\,(\varkappa/R)$ kOe · cm, and the overall current $\varkappa M\,8.5$ kA is proportional to the number of layers M. Here a series of concentric cathodes, involving alternating magnetic fluxes of the order $2\pi R\varkappa\,1.7$ kOe · cm, is necessary. The estimates made of the required external field at the cathode B_k are greatly exaggerated, since they do not take account of the axial self-field, which, in the beams under consideration, is directed the same way as the external field B_k, and substantially increases the total field at the cathode.

High currents are important for the realization of the states described above. A decrease in the relativistic factor $\varkappa$ and a proportional change in the currents do not alter the structure of the beam.

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